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The Hierarchical ϕ^4-Trajectory by Perturbation Theory in a Running Coupling and its Logarithm

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We compute the hierarchical ϕ^4-trajectory in terms of perturbation theory in a running coupling. In the three-dimensional case we resolve a singularity due to resonance of power counting factors in terms of logarithms of the running coupling. Numerical data are presented and the limits of validity explored. We also compute moving eigenvalues and eigenvectors on the trajectory as well as their fusion rules.

KEY WORDS: Hierarchical renormalization group; ϕ^4-trajectory; running coupling; logarithm; scaling; moving eigenvectors.

1. INTRODUCTION

In the block spin renormalization scheme of Wilson[19,13] renormalized theories come as renormalized trajectories of effective actions. Departing from a bare action, the renormalized trajectory is reached by an infinite iteration of block spin transformations. For this limit to exist the bare couplings have to be tuned as the number of block spin transformations is increased. Consider an asymptotically free model at weak coupling. There the point is to keep couplings under control which increase in value under a block spin transformation. Such couplings are called relevant. In weakly coupled models-they can be identified by naive power counting. This renormalization scheme has been beautifully implemented both within and beyond perturbation theory. We mention the work of Polchinski,[15] Gawedzki and Kupiainen,[10] Gallavotti,[9] and Rivasseau[17] as a guide to the extensive literature.

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The underlying picture of an ultraviolet asymptotically free model comes from thinking of the renormalized trajectory as the unstable manifold of a trivial fixed point. Although this picture has been behind block spin renormalization since the very beginning,[13] it has not been formalized yet to an approach free of a bare action. This paper is a contribution to fill this gap. It extends the analysis begun in refs. 20 and 21 in the context of renormalization group improved actions for the two-dimensional $O(N)$-invariant nonlinear σ-model. Here we will work it out for the ϕ^4-trajectory in the hierarchical approximation. The hierarchical model was invented by Dyson[4] and Baker[1] and has enjoyed the attention, e.g., of Bleher and Sinai,[2] Collet and Eckmann,[3] Koch and Wittwer,[14] Felder,[6] and Pordt.[16]

The ϕ^4-trajectory will be defined as a curve which departs the trivial fixed point in the ϕ^4-direction. Technically we perform a renormalized perturbation expansion in a running coupling. In the three-dimensional case we perform a perturbation expansion in a running coupling and its logarithm. The dynamical principle which proves to be strong enough to determine the trajectory at least in perturbation theory is stability under the renormalization group. By stability we mean that the trajectory is left invariant under a transformation as a set in the space of potentials. Recall that a renormalized action always comes together with a sequence of descendants generated by further block spin transformations. Even in the case of a discrete transformation this sequence will prove to consist of points on a continuous curve in the space of potentials which is stable under the renormalization group. It is the computation of this curve we address. The result is an iterative form of renormalized perturbation theory. Its closest relatives in the literature are the renormalized tree expansions of Gallavotti and collaborators.[9,11,12] A pedagogical presentation can be found in ref. 7.

Our expansion is, however, free of divergences piled up in standard perturbation theory by infinitely iterated renormalization group transformations from the very beginning. Surprisingly, we do not need to treat relevant and irrelevant couplings on a different footing. It will involve neither bare couplings nor renormalization conditions in the original sense. A renormalization group transformation in our approach translates to a transformation of the running coupling according to some β-function. We will consider in particular a choice of coordinate whose associated β-function is exactly linear. This idea has also appeared in ref. 5 and references therein. Renormalized perturbation theory furthermore will surprise us with a sequence of discrete poles at special rational dimensions. These poles will be traced back to certain resonance conditions on the scaling dimensions of powers of fields. In particular the case of three dimensions will be shown to be resonant. We will resolve the associated singularity by a double expansion in both the running coupling and its logarithm. The expansion will then be extended to the computation of moving eigenvectors in the sense of ref. 21 on the renormalized trajectory and their fusion rules. Finally, we perform a numerical test of our renormalized actions. As expected, they prove to work well in the small-field region. The extension of our program to full models is under way. A prototype with momentum space regularization has been developed in ref. 18.

2. HIERARCHICAL RENORMALIZATION GROUP

Hierarchical renormalization group transformations have been proposed in a number of forms. Their relationships have been investigated in ref. 14. The form used in ref. 10 is a theory of the nonlinear transformation

$$\mathcal{R}Z(\psi) = \left(\int d\mu_\gamma(\zeta)\, Z(L^{1-D/2}\psi + \zeta) \right)^{L^D} \tag{1}$$

on some space of Boltzmann factors $Z(\phi)$. It is equivalent to the transformations used in refs. 8 and 14. Furthermore, it is equivalent to that of ref. 3 in the case when $L^D = 2$. In the scalar theory ϕ is a single real field variable. We have that

$$d\mu_\gamma(\zeta) = (2\pi\gamma)^{-1/2} \exp\left(-\frac{\zeta^2}{2\gamma} \right) d\zeta \tag{2}$$

is the Gaussian measure on $\mathbb{R}$ with mean zero and covariance γ. The parameters of (1) are the Euclidean dimension D and the block scale L. The subspace of even Boltzmann factors $Z(-\phi) = Z(\phi)$ is stable under (1). We will restrict our attention to this subspace. Let the potential be given by $Z(\phi) = \exp(-V(\phi))$. The transformation for the potential is

$$\mathcal{R}V(\psi) = -L^D \log \left\{ \int d\mu_\gamma(\zeta) \exp[-V(L^{1-D/2}\psi + \zeta)] \right\} \tag{3}$$

The analysis below will be done in terms of the potential. The method will be perturbation theory. The question of stability bounds will not be addressed. Regarding mathematical aspects of (1) and (3), we refer to the work of Collet and Eckmann,[3] Gawedzki and Kupiainen,[10] and Koch and Wittwer.[14]

3. THE TRIVIAL FIXED POINT

Equation (3) has a trivial fixed point $V_*(\phi) = 0$. This fixed point is the hierarchical massless free field. The linearization of (3) at this trivial fixed point is given by

$$\mathscr{L}_{V_*} \mathscr{R}\mathcal{O}(\psi) = L^D \int d\mu_\gamma(\zeta)\, \mathcal{O}(L^{2-D/2}\psi + \zeta) \tag{4}$$

This linearization is diagonalizable. The eigenvectors are normal ordered products

$$:\phi^n:_{\gamma'} = \frac{\partial^n}{\partial j^n}\bigg|_{j=0} \exp\left(j\phi - \frac{j^2\gamma'}{2}\right) \tag{5}$$

with normal ordering covariance $\gamma' = (1 - L^{2-D})^{-1}\gamma$. The normal ordering covariance has been chosen in order to be invariant with respect to integration with $d\mu_\gamma$. Its singularity at $D = 2$ is an infrared singularity of the hierarchical massless free field in two dimensions. The eigenvalues are

$$\lambda_n = L^{D + n(1 - D/2)} \tag{6}$$

The eigenvalue of $:\phi^4:_{\gamma'}$ is $\lambda_4 = L^{4-D}$. The eigenvector $:\phi^4:_{\gamma'}$ is therefore relevant for $D < 4$, marginal for $D = 4$, and irrelevant for $D > 4$ dimensions. Perturbation theory can be used to compute corrections to (4) in a neighborhood of $V_*(\phi)$.

4. THE ϕ^4-TRAJECTORY

Let us define a curve $V(\phi, g)$ in the space of potentials parametrized by a local coordinate g. We call it the ϕ^4-trajectory. We expand the potential

$$V(\phi, g) = \sum_{n=0}^{\infty} V_{2n}(g) :\phi^{2n}:_{\gamma'} \tag{7}$$

in the base of eigenvectors (5). A natural coordinate in the vicinity of $V_*(\phi)$ is the ϕ^4-coupling defined by $V_4(g) = g$. Let us use it for a moment. Let the ϕ^4-trajectory then be the curve $V(\phi, g)$ defined by the following two conditions:

1. $V(\phi, g)$ is stable under $\mathscr{R}$. Then there exists a function $\beta(g)$ such that

$$\mathscr{R}V(\phi, g) = V(\phi, \beta(g)) \tag{8}$$

   The function $\beta(g)$ is of course coordinate dependent. With the ϕ^4-coupling as coordinate it is called the β-function.

2. $V(\phi, g)$ visits the trivial fixed point $V_*(\phi)$ at $g = 0$. The tangent to $V(\phi, g)$ at $V_*(\phi)$ is given by

$$\left.\frac{\partial}{\partial g}\right|_{g=0} V(\phi, g) = :\phi^4:_{\gamma} \tag{9}$$

   This condition is equivalent to $V_4(g) = g + O(g^2)$ together with $V_{2n}(g) = O(g^2)$, $n \neq 2$.

The ϕ^4-trajectory is the object of principal interest in massless ϕ^4-theory at weak coupling. We should mention that our analysis also applies to the noncritical ϕ^4-trajectory emerging from the high-temperature fixed point. The high-temperature renormalization group in the formulation of ref. 14 is in fact of the same form (3), but for a different scaling dimension, $-1 - D/2$ (instead of $1 - D/2$) of the scalar field and a trivial change of the fluctuation covariance. We omit a presentation of a perturbation theory for the high-temperature ϕ^4-trajectory for the sake of brevity. Let us only remark here that there are no resonances at the high-temperature fixed point.

5. PERTURBATION THEORY

The ϕ^4-trajectory can be computed by perturbation theory in g as the solution to (8) and (9). Potentials on the ϕ^4-trajectory are said to scale. A potential $V(\phi, g)$ is said to scale to order s in g if there exists a function

$$\begin{aligned} \beta(g) &= \beta^{(s)}(g) + O(g^{s+1}) \\ \beta^{(s)}(g) &= \sum_{r=1}^{s} b_r g^r \end{aligned} \tag{10}$$

such that

$$\begin{aligned} V(\phi, g) &= V^{(s)}(\phi, g) + O(g^{s+1}) \\ \mathscr{R}V^{(s)}(\phi, g) &= V^{(s)}(\phi, \beta(g)) + O(g^{s+1}) \end{aligned} \tag{11}$$

and

$$V^{(1)}(\phi, g) = g :\phi^4:_{\gamma'} \tag{12}$$

The scheme is to compute $\beta^{(s+1)}(g)$ and $V^{(s+1)}(\phi, g)$ given $\beta^{(s)}(g)$ and $V^{(s)}(\phi, g)$ to some order s. Let us explain it in some detail at the case of $D=4$ dimensions, block scale $L=2$, and covariance $\gamma=1$. Then the normal ordering covariance is $\gamma'=4/3$. Computing a block spin transformation (3), we speak of $V(\phi)$ as the bare potential and of $\mathscr{R}V(\phi)$ as the effective potential. The point of departure is (12). Anticipating the terms generated in $\mathscr{R}V^{(1)}(\phi, g)$ to second order in g, we make the ansatz

$$V^{(2)}(\phi, g) = c_0 g^2 + c_2 g^2 :\phi^2: + g :\phi^4: + c_6 g^2 :\phi^6: \tag{13}$$

The coefficients are determined by the condition that (11) be fulfilled to second order. Equation (13) is mapped to

$$\begin{aligned}\mathscr{R}V^{(2)}(\phi, g(g')) = {} & \left(16c_0 - \frac{5440}{9}\right) g'^2 + (4c_2 - 448)\, g'^2 :\phi^2: \\ & + g' :\phi^4: + \left(\frac{c_6}{4} - 2\right) g'^2 :\phi^6: + O(g'^3)\end{aligned} \tag{14}$$

Here the effective coupling, defined as the coefficient of $:\phi^4:$ in the effective potential, is given by

$$g'(g) = g - 60g^2 + O(g^3) \tag{15}$$

Comparing the effective potential as a function of the effective coupling with the bare potential as a function of the bare coupling, we conclude that

$$c_0 = \frac{1088}{27}, \qquad c_2 = \frac{448}{3}, \qquad c_6 = -\frac{8}{3} \tag{16}$$

on the ϕ^4-trajectory. The coefficients of the β-function (10) to this order are

$$b_1 = 1, \qquad b_2 = -60 \tag{17}$$

It follows that g is marginally irrelevant in four dimensions. This completes the first step. It is iterated in the obvious manner. The general form of the order-s approximation is

$$
\begin{aligned}
V^{(s)}(\phi, g) &= \sum_{n=0}^{s+1} c_{2n}^{(s)}(g) :\phi^{2n}: \\
c_{2n}^{(s)}(g) &= \sum_{r=2}^{s} c_{2n,r} g^r, \qquad n \leqslant 1 \\
c_4^{(s)}(g) &= g \\
c_{2n}^{(s)}(g) &= \sum_{r=n-1}^{s} c_{2n,r} g^r, \qquad n \geqslant 3
\end{aligned}
\tag{18}
$$

It includes all normal ordered products generated in the effective potential by (3) from (12) to order s in g. The iteration proceeds as above with the order-$(s+1)$ ansatz of the form (18). The condition (11) yields a system of linear equations for the order-$(s+1)$ coefficients. (To highest order the coefficients have no other choice.) This system has a unique solution: the ϕ^4-trajectory. Note that the coefficient b_{s+1} of the β-function is already determined by $V^{(s)}(\phi, g)$. For instance, (15) does not contain any of the coefficients in (13). The expansion can be computed to higher orders using computer algebra. To third order we find

$$
\begin{aligned}
c_0^{(3)}(g) &= \frac{1088}{27} g^2 - \frac{54784}{27} g^3 \\
c_2^{(3)}(g) &= \frac{448}{3} g^2 - \frac{497408}{27} g^3 \\
c_6^{(3)}(g) &= -\frac{8}{3} g^2 + 352 g^3 \\
c_8^{(3)}(g) &= \frac{32}{3} g^3
\end{aligned}
\tag{19}
$$

together with

$$
\beta^{(3)}(g) = g - 60g^2 + 8880g^3 \tag{20}
$$

Let us remark that the perturbation coefficients (10) and (11) come with alternating signs. The coefficients show a frightening increase in absolute value with the order in g. The full series is not expected to converge. Note that the coefficients look better when g is replaced by $g/4!$.

6. RESONANCES

We can apply the above scheme to compute the ϕ^4-trajectory in other than four dimensions. The solution is again of the form (18). We do however, encounter a new phenomenon. The improvement coefficients $c_{2n}^{(s)}$ exhibit poles at certain discrete points as functions of the dimension parameter D. We call this phenomenon resonance because it can be traced back to the fact that certain scaling parameters become powers of one another at these points. This happens in particular in three dimensions. Let us consider the transformation (3) with block scale $L=2$ and covariance $\gamma=1$, but this time with arbitrary value of D. We express the dependence on D in terms of a variable $\alpha=L^D$. To third order in g the improvement coefficients are given by the rational functions

$$\begin{aligned}
c_0^{(3)} &= \frac{12\alpha^3(\alpha+4)(\alpha^2+16)}{(\alpha^3-256)(\alpha-4)^3}g^2 \\
&\quad -\frac{288\alpha^4(\alpha^5+32\alpha^3+512\alpha^2+4096)(\alpha+4)^2}{(\alpha+8)(\alpha-8)(\alpha^2+64)(\alpha^3-256)(\alpha-4)^4}g^3 \\
c_2^{(3)} &= \frac{48\alpha^2(\alpha^2+4\alpha+16)}{(\alpha-8)(\alpha+8)(\alpha-4)^2}g^2 \\
&\quad -\frac{384\alpha^3(7\alpha^5+46\alpha^4+288\alpha^3+1728\alpha^2-1024\alpha-22528)}{(\alpha+8)(\alpha-8)(\alpha^3-1024)(\alpha-4)^3}g^3 \\
c_6^{(3)} &= -\frac{8}{3}g^2-\frac{576\alpha(\alpha+6)}{(\alpha-4)(\alpha-64)}g^3 \\
c_8^{(3)} &= \frac{32}{3}g^3
\end{aligned} \tag{21}$$

The β-function to this order in g is given by the function

$$\begin{aligned}
\beta^{(3)}(g) &= \frac{16}{\alpha}g-\frac{576(\alpha+4)}{\alpha(\alpha-4)}g^2 \\
&\quad +\frac{256(215\alpha^4+1400\alpha^3-10128\alpha^2)}{\alpha(\alpha-8)(\alpha+8)(\alpha-4)^2}g^3 \\
&\quad -\frac{256(95744\alpha+355328)}{\alpha(\alpha-8)(\alpha+8)(\alpha-4)^2}g^3
\end{aligned} \tag{22}$$

Singularities appear at positive dimensions

	g^2			g^3	
α	4	$256^{1/3}$	8	$1024^{1/3}$	64
D	2	8/3	3	10/3	6

(23)

As one goes to higher orders in g, more and more poles show up in (21) and (22). Let us have a closer look at the three-dimensional pole to second order in g. Inserting (13) into (3) at three dimensions, we find

$$\mathscr{R}V^{(2)}(\phi, g(g')) = (2c_0 - 360) + (c_2 - 336)\, g'^2 :\phi^2: + g' :\phi^4: + \left(\frac{c_6}{4} - 2\right) g'^2 :\phi^6: + O(g'^3) \tag{24}$$

The parameters of (3) are $D = 3$, $L = 2$, and $\gamma = 1$. The normal ordering covariance is $\gamma' = 2\gamma$. The β-function to this order is given by

$$g'(g) = 2g - 216g^2 + O(g^3) \tag{25}$$

The $:\phi^4:$-coupling is therefore relevant in three dimensions. From (24) we would conclude that

$$c_0 = 360, \qquad c_6 = -8/3 \tag{26}$$

But there exists no solution to the equation for c_2 (besides infinity). It follows that the $:\phi^2:$-coupling cannot be written in terms of a power series in the $:\phi^4:$-coupling on the ϕ^4-trajectory in three dimensions. The point is that the $:\phi^2:$-coupling flows like

$$-336n2^{2n} = -\frac{336}{\log(2)} \log(g)\, g^2 \tag{27}$$

with $g = 2^n$ upon iteration of (24). This suggests a double expansion in both g and $\log(g)$.

7. LINEAR β-FUNCTION

So far we have used the ϕ^4-coupling as coordinate for the ϕ^4-trajectory. It was defined by the condition $V_4(g) = g$ following (7). This coordinate leads to unnecessary complications when dealing with its logarithm. A better coordinate which is also interesting by itself is the linear coordinate defined by the condition that the β-function be exactly given by

$$\beta(g) = L^{4-D} g \tag{28}$$

on the ϕ^4-trajectory. In terms of this linear β-function the transformation on the ϕ^4-trajectory looks exactly like the linearized renormalization group. The definitions (8) and (9) remain untouched by (28). Using the expansion with the linear β-function, we find that

$$V_4(g) = g + \sum_{r=2}^{\infty} V_{4,r} g^r \tag{29}$$

also becomes a computable power series. The strategy is the same as above. For the case of $L = 2$ and $\gamma = 1$ the ϕ^4-trajectory is given by

$$\begin{aligned}
V_0^{(3)} &= \frac{12\alpha^3(\alpha+4)(\alpha^2+16)}{(\alpha^3-256)(\alpha-4)^3} g^2 \\
&\quad + \frac{576\alpha^4(\alpha^3+8\alpha^2+8\alpha+256)(\alpha+4)^2}{(\alpha-16)(\alpha+8)(\alpha-8)(\alpha^2+64)(\alpha-4)^4} g^3 \\
V_2^{(3)} &= \frac{48\alpha^2(\alpha^2+4\alpha+16)}{(\alpha-8)(\alpha+8)(\alpha-4)^2} g^2 \\
&\quad + \frac{768\alpha^3(\alpha^6+69\alpha^5+368\alpha^4-2880\alpha^3)}{(\alpha-16)(\alpha+8)(\alpha-8)(\alpha^3-1024)(\alpha-4)^3} g^3 \\
&\quad + \frac{768\alpha^3(-22528\alpha^2-144384\alpha-475136)}{(\alpha-16)(\alpha+8)(\alpha-8)(\alpha^3-1024)(\alpha-4)^3} g^3 \\
V_4^{(3)} &= g + \frac{36\alpha(\alpha+4)}{(\alpha-4)(\alpha-16)} g^2 \\
&\quad - \frac{16\alpha^2(53\alpha^5-3336\alpha^4-24752\alpha^3)}{(\alpha+8)(\alpha-8)(\alpha+16)(\alpha-16)^2(\alpha-4)^2} g^3 \\
&\quad - \frac{16\alpha^2(149248\alpha^2+1342464\alpha+5685248)}{(\alpha+8)(\alpha-8)(\alpha+16)(\alpha-16)^2(\alpha-4)^2} g^3 \\
V_6^{(3)} &= -\frac{8}{3} g^2 - \frac{384\alpha(2\alpha^2-45\alpha-272)}{(\alpha-4)(\alpha-16)(\alpha-64)} g^3 \\
V_8^{(3)} &= \frac{32}{3} g^3
\end{aligned} \tag{30}$$

To each order of perturbation theory we find a system of linear equations which has a unique solution. The coefficients are again rational functions in $\alpha = L^D$ with poles at resonant dimensions. Note that (30) has an additional pole at $D=4$ as compared with (21). In four dimensions (28) becomes the identity and (8) a fixed-point equation. Resonances are now easily understood. To order m in g the transformation (3) acts as

$$\begin{aligned} cg^m :\phi^{2n}: &\mapsto L^{D+n(2-D)} cg^m :\phi^{2n}: \\ &= L^{D+n(2-D)-m(4-D)} c\beta(g)^m :\phi^{2n}: \end{aligned} \tag{31}$$

A resonance thus occurs if

$$D + n(2-D) - m(4-D) = 0 \tag{32}$$

In the case of $D=3$ dimensions this condition becomes

$$3 - n - m = 0 \tag{33}$$

Since $m \geqslant 2$ in this business we find only two resonant terms (n, m) in three dimensions: $(1, 2)$ and $(0, 3)$. The former is a mass resonance, the latter a vacuum resonance. The resonant dimensions are rational and given by

$$D = \frac{4m - 2n}{1 - n + m} \tag{34}$$

In particular table (23) is immediately reproduced. An interesting variation of the linear β-function consists in replacing (28) by

$$\begin{aligned} \beta(g) &= b_1 g + b_2 g^2 \\ b_1 &= L^{4-D} \\ b_2 &= 36 L^{4-D} \frac{L^2 + L^D}{L^2 - L^D} \end{aligned} \tag{35}$$

truncating the β-function (8) to second (or more generally to any fixed) order of perturbation theory in the ϕ^4-coupling g. This β-function has a nontrivial fixed point at finite g for $D < 4$. It is conceivable that our expansion is valid at this fixed point at least for dimensions $D = 4 - \varepsilon$. The coefficient b_1 is universal, whereas b_2 is a matter of choice. It can be used to tune the fixed point to small couplings. We will not pursue this line of thought further at this instant. With the linear β-function (28) the nontrivial fixed point is moved to infinite coupling. It is, however, best suited for the double expansion in both g and $\log(g)$.

8. PERTURBATION THEORY IN g AND $\log(g)$

Let us consider the ϕ^4-trajectory in $D=3$ dimensions, again with $L=2$ and $\gamma=1$. The problem that c_2 cannot be determined in (24) such that it remains invariant to second order can be cured as follows. We use the linear β-function defined by (28) and expand the potential in both g and

$$\kappa = \log(g) \tag{36}$$

In the expansion we treat κ as an independent variable having the same order as g^0. It appears in the combination κg^2, which is really the small parameter. To second order in g^2 we can replace (13) by the ansatz

$$V^{(2)}(\phi, g) = c_0 g^2 + (c_2 + c_{2,1}\kappa)\, g^2 :\phi^2: + (g + c_4 g^2) :\phi^4: + c_6 g^2 :\phi^6: \tag{37}$$

Here we have left out all terms which anyway turn out to be zero on the ϕ^4-trajectory. In terms of $g' = 2g$ and $\kappa' = \kappa + \log(2)$ the ansatz (37) is mapped by (3) to

$$\begin{aligned}\mathscr{R}V^{(2)}(\phi, g(g')) = (2c_0 - 360)\, g'^2 \\ &+ (c^2 - \log(2)\, c_{2,1} - 336 + c_{2,1}\kappa')\, g'^2 :\phi^2: \\ &+ \left(g' + \left(\frac{c_4}{2} - 54\right) g'^2\right) :\phi^4: \\ &+ \left(\frac{c_6}{4} - 2\right) g'^2 :\phi^6: + O(g'^3)\end{aligned} \tag{38}$$

As a consequence, (37) reproduces its form up to a change of the running coupling (28) if and only if

$$c_0 = 360, \qquad c_4 = -108, \qquad c_6 = -\frac{8}{3}, \qquad c_{2,1} = -\frac{336}{\log(2)} \tag{39}$$

The parameter c_2 is free. To second order in g we find a one-parameter family of solutions to (8) and (9). The free parameter is associated with the mass resonance (1, 2) of (33). One immediately anticipates another free parameter to third order in g coming with the vacuum resonance (0, 3). This is indeed the case. The general solution of (8) and (9) to third order in g is given by

$$
\begin{aligned}
V^{(3)}(\phi, g) = {} & 360g^2 + \frac{54432}{\log(2)} g^3\kappa + c_0 g^3 \\
& + \left(c_2 g^2 - \frac{336}{\log(2)} g^2\kappa + (116928 - 36c_2) g^3 + \frac{12096}{\log(2)} g^3\kappa \right) :\phi^2: \\
& + \left(g - 108g^2 + \left(17520 - \frac{8}{3} c_2 \right) g^3 + \frac{896}{\log(2)} g^3\kappa \right) :\phi^4: \\
& + \left(- \frac{8}{3} g^2 + 864g^3 \right) :\phi^6: \\
& + \frac{32}{3} g^3 :\phi^8:
\end{aligned}
\tag{40}
$$

including c_0 and c_2 as free parameters. Thereafter we have no further free parameters in the higher order coefficients. Thus we have to supplement the definition of the ϕ^4-trajectory given by (8) and (9) by two additional conditions on c_0 and c_2 in three dimensions to single out a curve in the space of interactions. We choose

$$
c_0 = 0, \qquad c_2 = 0 \tag{41}
$$

With this choice the perturbation theory has a minimal number of vertices. The conditions (41) can be thought of as additional renormalization conditions.

At higher orders the scheme explained above iterates. The general form of the potential at order s is

$$
\begin{aligned}
V^{(s)}(\phi, g) &= \sum_{n=0}^{s+1} V^{(s)}_{2n}(g) :\phi^{2n}: \\
V^{(s)}_{2n}(g) &= \sum_{r=2}^{s} \sum_{t=0}^{[r/2]} V_{2n,r,t}\, g^r \kappa^t, \qquad n \leqslant 1 \\
V^{(s)}_{4}(g) &= g + \sum_{r=2}^{s} \sum_{t=0}^{[r/2]} V_{4,r,t}\, g^r \kappa^t \\
V^{(s)}_{2n}(g) &= \sum_{r=n-1}^{s} \sum_{t=0}^{[r/2]} V_{2n,r,t}\, g^r \kappa^t, \qquad n \geqslant 3
\end{aligned}
\tag{42}
$$

Here the third-order coefficients are given by (40) and, for instance, (41). To each further order of perturbation theory we meet a system of linear equations possessing a unique solution for the coefficients in (42): the ϕ^4-trajectory in terms of the double expansion. Recall that κ should be substituted by $\log(g)$ in (42).

9. NUMERICAL CALCULATION OF THE RENORMALIZED TRAJECTORY

Hierarchical renormalization group flows can also be computed using standard numerical methods. Let us perform a numerical investigation of the transformation (3) in order to determine the limits of validity of the expansion (42) in g and $\log(g)$. Furthermore, we want to investigate the large-field behavior of our expansion. We choose the following numerical setup. To calculate the transformation (3) iteratively, we sample the potential V at N equidistant points between 0 and $\phi_{\max}$. Then we perform a cubic spline interpolation and integrate using standard NAG library functions. To reduce the error due to boundary effects at $\phi = \phi_{\max}$ we always choose $\phi_{\max}$ so that $V(\phi_{\max}) = V_{\max}$ with $V_{\max}$ large enough; for example, $V_{\max} = 20$ will do. For $\phi > \phi_{\max}$ we set

$$V(\phi) = V_{\mathrm{HT}}(\phi - a) - b \tag{43}$$

and choose a and b so that the first derivative of V at $\phi_{\max}$ is continuous. Here by V_{HT} we mean the quadratic high-temperature fixed point of (3). Field asymptotics have been investigated by Koch and Wittwer in their work on the nontrivial double-well fixed point.[14] That means that we supplement our numerical calculation with the expectation that the potentials on the renormalized trajectory have HT-like asymptotic behavior. We let the fluctuation field ζ vary between $-\zeta_{\max}$ and $\zeta_{\max}$ and choose $\zeta_{\max} = 20$. This produces errors which can be neglected. As in the case of the expansion, we restrict our attention to the space of symmetric potentials which is invariant under (3). All potentials were calculated with $L = 2$, $D = 3$, $\gamma = 1$, and $N = 401$. To determine the renormalized trajectory according to our definition in Section 4 supplemented by the condition $c_2 = 0$ (see Section 8), we proceed as follows. We start with a bare potential

$$\begin{aligned} V = &\left(\frac{336}{\log(2)} g_0^2 \kappa + 116928 g_0^3 + \frac{12096}{\log(2)} g_0^3 \kappa\right) :\phi^2: \\ &+ \left(g_0 - 108 g_0^2 + 17520 g_0^3 + \frac{896}{\log(2)} g_0^3 \kappa\right) :\phi^4: \\ &+ \left(-\frac{8}{3} g_0^2 + 864 g_0^3\right) :\phi^6: \\ &+ \frac{32}{3} g_0^3 :\phi^8: \end{aligned} \tag{44}$$

and choose g_0 to be a sufficiently small number, which means that an iteration starting with $L^{-m}g_0$ would yield the same trajectory if $m \in \mathbb{N}$. We have chosen $g_0 = 10^{-6}$. It is convenient to normalize the potential with the condition $V(\phi = 0) = 0$. The numerically determined potential will be referred to as exact in the following. We perform 12 iterations of the transformation (3). The analogous perturbative potentials can be found by solving the equation

$$g_0 = V_4(\tilde{g}_0) \tag{45}$$

for $\tilde{g}_0$ (which gives approximately 10^{-6}, of course) and then follow the renormalization group flow to

$$L^n \tilde{g}_0, \qquad n = 0,..., 13 \tag{46}$$

For the calculation of the perturbative potentials we used the seventh order of our perturbation expansion. The comparison between the exact and the perturbative potentials can be seen in the following figures. The exact potentials correspond to the continuous lines, the perturbative potentials correspond to the dashed lines. In Fig. 1 one can see that after eight renormalization group steps the exact and the perturbative data are nearly the same. If, however, the number of renormalization group steps exceeds nine,

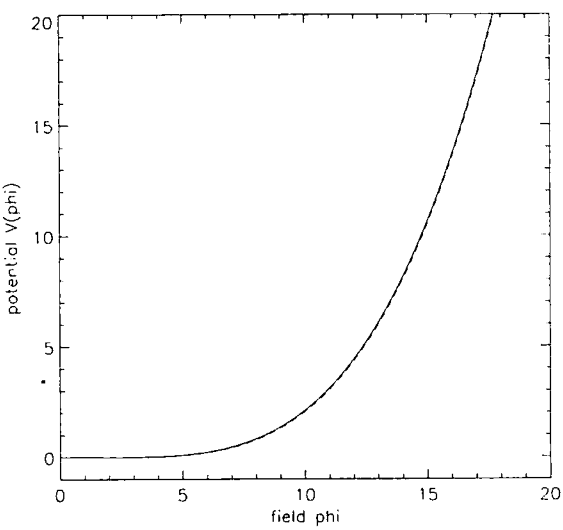

Fig. 1. Exact and perturbative potential after eight renormalization group steps. Perturbative data correspond with dashed lines. We started the iteration near the trivial fixed point. Detailed information can be found in the text.

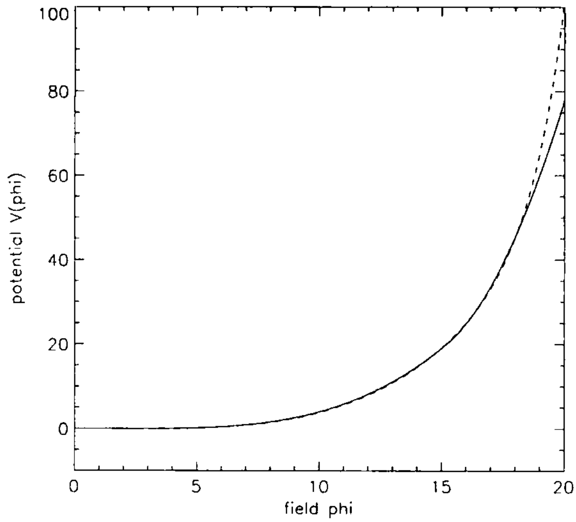

Fig. 2. Exact and perturbative potential after nine renormalization group steps. Above $V(\phi)=20$ we assumed HT-like behavior. There are deviations due to the wrong large-field behavior of our expansion.

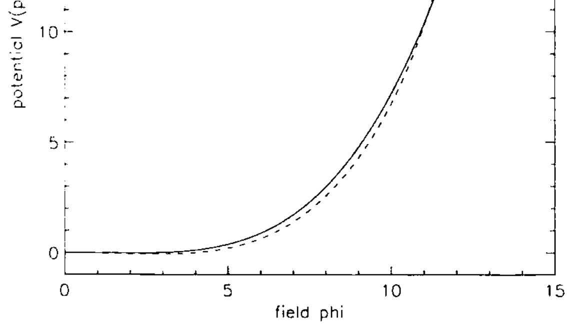

Fig. 3. Exact and perturbative potential after ten renormalization group steps. The perturbation expansion is only valid in the small-field region. This figure illustrates the borderline of the region of validity of our expansion in g and $\log(g)$.

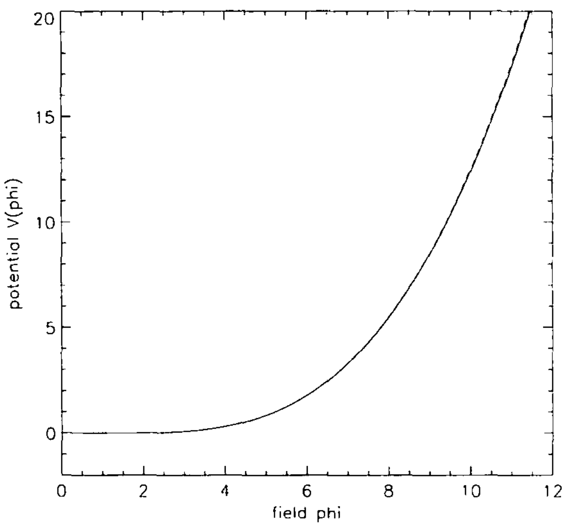

Fig. 4. Exact potential and Padé approximant of the perturbative potential after 11 renormalization group steps. Both curves coincide because the Padé approximant has the right large-field behavior.

there is a clear digression for $\phi \geqslant 18$. This can be seen in Fig. 2. That means that the large-field behavior of the perturbative expansion is wrong, since the asymptotic behavior of the exact potential is quadratic. If the number of renormalization group steps is bigger than ten, the perturbative expansion is only valid for very small fields. We therefore conclude that our expansion is no longer valid for $g \gtrsim 10^{-3}$. The borderline is illustrated in Fig. 3. One can, however, improve the perturbative data if one works with Padé approximants. In Fig. 4 we compare the (9, 7) Padé approximant of our perturbation series in the variable ϕ (which has the expected asymptotic behavior) with the numerical data after 11 renormalization group steps. Amazingly enough, both curves coincide. We will not go in details here. Let us only mention that some Padé approximants develop additional unphysical poles at real values of ϕ.

10. OBSERVABLES

The potential itself is not the only object of interest on the renormalized trajectory. Its infinitesimal neighborhood is also of physical significance. We can study this neighborhood in terms of certain generalized eigenvectors of the linearized renormalization group. At the trivial fixed point these eigenvectors are simply normal ordered products and their eigenvalues determine their scaling dimensions and (trivial) critical exponents.

Away from the fixed point we can deform these normal ordered products with a twofold intent. First, a renormalization group transformation always comes together with a transformation of local operators preserving expectation values. Our deformations are particular in the sense that they are eigenvectors of this transformation and hence only get multiplied by a number. Second, one may ask at what pace one departs from the renormalized trajectory (and hence criticality) upon infinitesimal perturbations. The leading eigenvalue determines this pace of departure. In the hierarchical approximation a local observable is a function $\mathcal{O}(\phi)$, $\phi \in \mathbb{R}$. Observables are transformed according to the linearized renormalization group. The linearization $\mathscr{L}_V \mathscr{R} \mathcal{O}$ of the transformation (3) in the direction $\mathcal{O}$ at the potential V is given by

$$\mathscr{L}_V \mathscr{R} \mathcal{O}(\psi) = \frac{\partial}{\partial z}\bigg|_{z=0} \mathscr{R}(V + z\mathcal{O})(\psi) \tag{47}$$

Written out explicitly, this gives

$$\mathscr{L}_V \mathscr{R} \mathcal{O}(\psi) = \frac{\int d\mu_\gamma(\zeta)\, \mathcal{O}(L^{1-D/2}\psi + \zeta) \exp(-V(L^{1-D/2}\psi + \zeta))}{\int d\mu_\gamma(\zeta) \exp(-V(L^{1-D/2}\psi + \zeta))} \tag{48}$$

after division by L^D. A local observable will be called a running eigenvector on the renormalized trajectory if it satisfies the invariance condition

$$\mathscr{L}_{V_{\mathrm{RT}}} \mathscr{R} \mathcal{O}(\phi, g) = e(\beta(g))\, \mathcal{O}(\phi, \beta(g)) \tag{49}$$

$e(g)$ will be called the corresponding running eigenvalue. This definition is not unambiguous. It still allows the freedom to multiply an observable with a function $f(g)$ changing both the observable and more, annoying, also the eigenvalue. The origin of this freedom is, however, simply that we can choose a different normalization in the tangent space at each point over the renormalized trajectory. Thus we have to supplement a normalization condition. Our observables will be indexed by an integer n. The nth observable will be a deformation of $:\phi^n:_\gamma$. We then choose the normalization that the coefficient in front of $:\phi^n:_\gamma$ of the nth observable in a normal ordered basis is one. We can then identify the eigenvalue as this coefficient of the effective observable. The notion of running eigenvectors and eigenvalues was introduced in ref. 21. Different local observables are distinguished by their values at the trivial fixed point. Starting from a normal ordered monomial, we can apply perturbation theory to compute corrections in g. The result is a moving frame in the tangent space over the renormalized trajectory. Consider specifically the eigenvector $:\phi^n:_\gamma$ at the trivial fixed point in Section 3. The corresponding running observable to the order s of perturbation theory will be called $\mathcal{O}_n^{(s)}(\phi, g)$. The initial condition is $\mathcal{O}_n^{(s)}(\phi, 0) = :\phi^n:_{\gamma'}$. The corresponding running eigenvalue to this approximation will be denoted by $e_n^{(s)}(g)$. Let us explain the perturbative scheme in some detail by calculating $\mathcal{O}_2^{(1)}(\phi, g)$ in three dimensions, using the linear β-function. To first order of perturbation theory for this observable we find

$$\mathcal{O}_2^{(1)}(\phi, g) = c_{4,0}\, g :\phi^4: + :\phi^2: + (c_{0,0} + c_{0,1}\kappa)\, g \tag{50}$$

This form is reproduced under the renormalization group to this order. The effective observable turns out to be

$$\begin{aligned}\mathcal{L}_{V_{\mathrm{RT}}}\mathcal{R}\mathcal{O}_2^{(1)}(\phi, g) = {} & (\tfrac{1}{8}c_{4,0} - 1)\, g :\phi^4: \\ & + (\tfrac{1}{2} - 9g) :\phi^2: \\ & + \{\tfrac{1}{2}c_{0,0} + \tfrac{1}{2}c_{0,1}[\kappa - \ln(2)]\}\, g\end{aligned} \tag{51}$$

Therefore the running eigenvalue $e_2^{(1)}(g)$ is, to this order,

$$e_2^{(1)}(g) = \tfrac{1}{2} - 9g \tag{52}$$

Division by $e_2^{(1)}(g)$ yields an effective observable of the form

$$\begin{aligned}\mathcal{O}_{2,\mathrm{eff}}^{(1)} = {} & (\tfrac{1}{4}c_{4,0} - 2)\, g :\phi^4: \\ & + :\phi^2: \\ & + \{c_{0,0} + c_{0,1}[\kappa - \ln(2)]\}\, g\end{aligned} \tag{53}$$

after rescaling g. Invariance requires the coefficients to take the values

$$c_{4,0} = -8/3, \qquad c_{0,1} = 0 \tag{54}$$

The value of $c_{0,0}$ is not constrained. Therefore we have an additional renormalization condition for our observable at this order. To derive the general resonance condition for observables consider the contribution $cg^l :\psi^m:$ to $\mathcal{O}_n$. It is mapped to

$$cL^{m(1-D/2) + l(D-4) - n(1-D/2)} g^l :\psi^m: \tag{55}$$

after division by $e_n(g)$. Observable resonances therefore occur at rational dimensions

$$D = \frac{8l + 2n - 2m}{2l - m + n} \tag{56}$$

or

$$m = \frac{2l(4-D)}{2-D} + n \tag{57}$$

For the case $D = 3$ this means that we have extra renormalization conditions when

$$m = n - 2l \tag{58}$$

For instance, the observable $\mathcal{O}_4$ requires two conditions.

The following list displays the first three observables using the parameters $L = 2$, $D = 3$, and $\gamma = 1$. Free parameters are set to zero. We have

$$\begin{aligned}
\mathcal{O}_0^{(3)}(\phi, g) &= 1 \\
\mathcal{O}_1^{(3)}(\phi, g) &= :\phi:_{\gamma'} \\
&\quad + \left(-\frac{4}{3} g + 144 g^2 - 896 \frac{\kappa g^3}{\ln(2)} - 23360 g^3 \right) :\phi^3:_{\gamma'} \\
&\quad + \left(\frac{16 g^2}{3} - 1728 g^3 \right) :\phi^5:_{\gamma'} \\
&\quad - \frac{256}{9} :\phi^7:_{\gamma'} \\
\mathcal{O}_2^{(3)}(\phi, g) &= -960 g^2 + \frac{2016 \kappa g^2}{\ln(2)} - 632448 g^3 \\
&\quad + :\phi^2:_{\gamma'} \\
&\quad + \left(-\frac{8g}{3} + 432 g^2 - \frac{1792 g^3 \kappa}{\ln(2)} - \frac{483072 g^3}{5} \right) :\phi^4:_{\gamma'} \\
&\quad + \left(\frac{112 g^2}{9} - \frac{164160 g^3}{31} \right) :\phi^6:_{\gamma'} \\
&\quad - \frac{640 g^3}{9} :\phi^8:_{\gamma'}
\end{aligned} \tag{59}$$

The corresponding eigenvalues are

$$e_0^{(3)} = 1$$

$$e_1^{(3)} = \frac{\sqrt{2}}{2} + \frac{84\sqrt{2}\,\kappa g^2}{\ln(2)} + 28\sqrt{2}\,g^2 - \frac{3528\sqrt{2}\,\kappa g^3}{\ln(2)} - 34608\sqrt{2}\,g^3 \tag{60}$$

$$e_2^{(3)} = \frac{1}{2} - 9g + \frac{168\kappa g^2}{\ln(2)} + 1232 g^2 - \frac{17136 g^3 \kappa}{\ln(2)} - 287736 g^3$$

In the following section on fusion rules and in the numerical section we have used seventh-order approximants. They will not be displayed, because of lack of space.

11. FUSION RULES

The computation correlation functions of local observables requires the knowledge of their fusion rules. The fusion rules of two observables $\mathcal{O}_n(\phi, g)$ and $\mathcal{O}_m(\phi, g)$ are defined by

$$\mathcal{O}_n(\phi, g)\,\mathcal{O}_m(\phi, g) = \sum_{l=0}^{\infty} N^l_{nm}(g)\,\mathcal{O}_l(\phi, g) \tag{61}$$

The coefficient N^0_{nm} induces a symmetric bilinear form $\langle\ ,\ \rangle$ on the space of observables by

$$\langle \mathcal{O}_n(\phi, g), \mathcal{O}_m(\phi, g)\rangle := N^0_{nm}(g) \tag{62}$$

In the thermodynamic limit only this overlap of their product with the constant term survives. See ref. 21 for further details. The perturbation expansion for the fusion rules takes the form

$$N^{l(s)}_{nm}(g) = \sum_{k=0}^{s} N^l_{nm(k)}\, g^k \tag{63}$$

To zeroth order we recover the fusion rules for normal ordered products. Consider parameter values $D=3$, $\gamma=1$ with normal ordering covariance $\gamma'=2$. To second order we then find

$$\begin{aligned}\mathcal{O}_1^{(2)}(\phi, g)\,\mathcal{O}_1^{(2)}(\phi, g) = {} & \left(2 + g^2\left(-2016\,\frac{\kappa}{\ln(2)} + \frac{3136}{3}\right)\right)\mathcal{O}_0^{(2)}(\phi, g) \\ & + (1 - 16g + 1856g^2)\,\mathcal{O}_2^{(2)}(\phi, g) \\ & - 48g^2\mathcal{O}_4^{(2)}(\phi, g)\end{aligned}$$

$$\begin{aligned}\mathcal{O}_1^{(2)}(\phi, g)\,\mathcal{O}_2^{(2)}(\phi, g) = \Bigg(& 4 + g\left(168\,\frac{\kappa}{\ln(2)} - 32\right) \\ & + g^2\left(8736\,\frac{\kappa}{\ln(2)} - \frac{165856}{3}\right)\Bigg)\,\mathcal{O}_1^{(2)}(\phi, g) \\ & + \left(1 - 32g + g^2\left(224\,\frac{\kappa}{\ln(2)} + 5504\right)\right)\mathcal{O}_3^{(2)}(\phi, g) \\ & - 96g^2\mathcal{O}_5^{(2)}(\phi, g)\end{aligned} \tag{64}$$

$$\begin{aligned}\mathcal{O}_2^{(2)}(\phi, g)\,\mathcal{O}_2^{(2)}(\phi, g) = & \left(8 - 720g + g^2\left(-179424\,\frac{\kappa}{\ln(2)} + \frac{169472}{3}\right)\right)\mathcal{O}_0^{(2)}(\phi, g) \\ & + \Bigg(8 + g\left(672\,\frac{\kappa}{\ln(2)} - 256\right) \\ & + 8g^2\left(69888\,\frac{\kappa}{\ln(2)} - \frac{917312}{3}\right)\Bigg)\,\mathcal{O}_2^{(2)}(\phi, g) \\ & + \left(1 - 64g + g^2\left(1792\,\frac{\kappa}{\ln(2)} + 14720\right)\right)\mathcal{O}_4^{(2)}(\phi, g) \\ & - 192g^2\mathcal{O}_6^{(2)}(\phi, g)\end{aligned}$$

Free renormalization parameters have been put to zero. Perturbation theory leaves us with a deformed normal ordered fusion algebra. Its structure will not be investigated here.

12. NUMERICAL CALCULATION OF THE OBSERVABLES AND EIGENVALUES

Similar to the potentials on the renormalized trajectory, we are now going to calculate the eigenvalues and observables of our theory in three dimensions numerically. The aim is again the determination of the region of validity of our perturbative expansion in g and $\log(g)$ now for the eigenvalues and observables. To this end we restrict the action of the linearized renormalization group transformation (48) to the finite-dimensional space of observables which is spanned by

$$\phi^m, \qquad 0 \leqslant m \leqslant M \tag{65}$$

After performing an expansion (which is most conveniently done by differentiation), we get a finite-dimensional representation matrix L of the linearized renormalization group transformation

$$\mathscr{L}_{V_{RT}}\mathscr{R}\phi^i = \sum_{j=0}^{M} L_{i,j}\phi^j \tag{66}$$

Now we are able to calculate the eigenvalues and observables of L. Because the matrix L is of course only an approximation of the transformation (48) we have to choose M big enough to get correct results. We have chosen $M = 8$ and expect that the first four eigenvalues and eigenvectors do not suffer from big errors due to the trucation. In Fig. 5 we plot the four largest eigenvalues of transformation (48) against the number of renormalization group steps. The crosses correspond to the exact (i.e., numerical) calculation, whereas the boxes correspond to our perturbation expansion. We start at the perturbative potential at $g_0 = 10^{-6}$, which means near the trivial fixed point. Then we follow the renormalized trajectory, performing numerical and perturbative renormalization group steps. After six renormalization group steps, i.e., at $g = 2^6 g_0$ perturbatively (compare Section 7), one can see the first small deviations between the perturbative and the exact eigenvalues. After ten renormalization group steps (i.e., at $g = 1.0 \times 10^{-3}$) there is a clear distinction between both. We recover and sharpen the former result that for $g \gtrsim 10^{-3}$ our expansion cannot be said to be valid any longer because of nonperturbative effects.

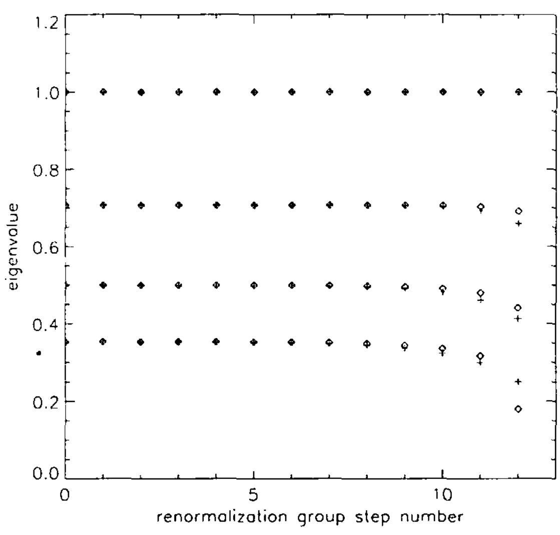

Fig. 5. Exact (crosses) and perturbative (boxes) eigenvalues of transformation (48) against the number of renormalization group steps. The iteration starts near the trivial fixed point.

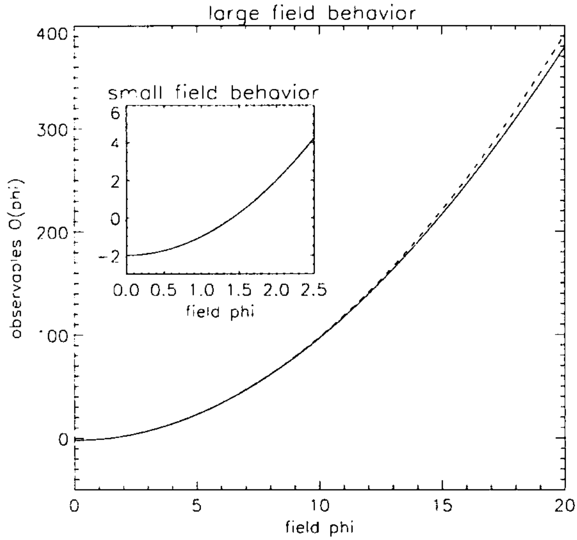

Fig. 6. Exact and perturbative $\mathcal{C}'_2(\phi)$ after four renormalization group steps. In the small-field region our expansion performs very well.

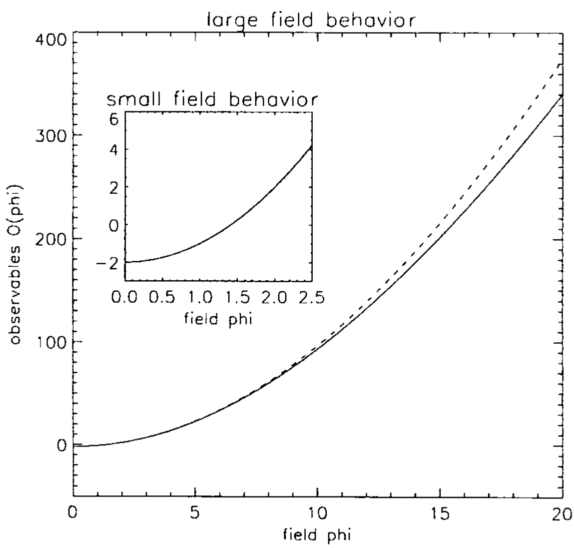

Fig. 7. Exact and perturbative $\mathcal{C}'_2(\phi)$ after six renormalization group steps. Of course the deviations at large values of g are due to the wrong large-field behavior of the perturbation expansion.

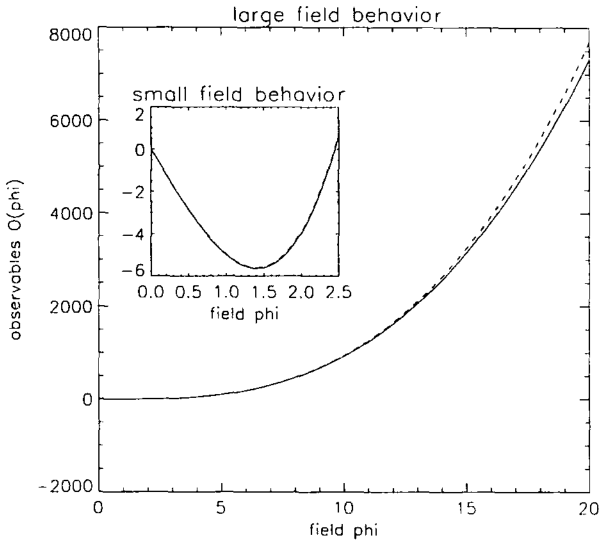

Fig. 8. Exact and perturbative $\mathcal{O}_3(\phi)$ after four renormalization group steps.

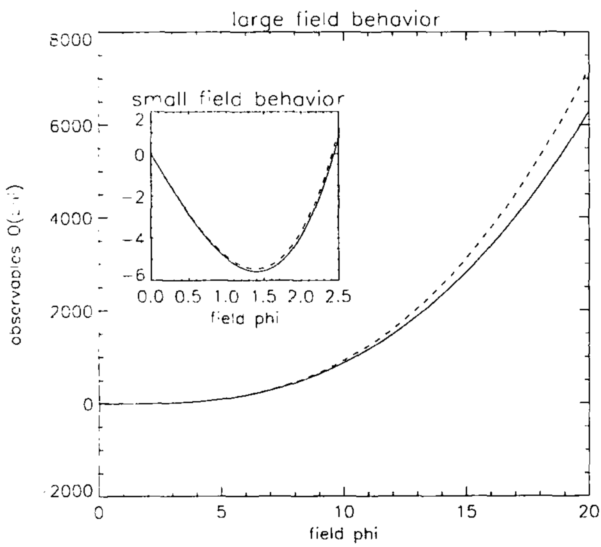

Fig. 9. Exact and perturbative $\mathcal{O}_3(\phi)$ after six renormalization group steps.

In Figs. 6–9 we plot the exact and perturbative observables $\mathcal{O}_2$ and $\mathcal{O}_3$ after four and after six renormalization group steps, respectively. Here the perturbative data correspond to the dashed lines. For large fields we have deviations for both observables which are to some extent due to the truncation of the transformation (48). The small-field behavior has been replotted in the insets in order to illustrate the more significant influence of the truncation on the observable $\mathcal{O}_3$ as well as to illustrate the right behavior at $\phi = 0$ of our calculations. Here we just recover the aproximate normal ordering near the trivial fixed point. Because of nonperturbative effects, we cannot reach the nontrivial fixed point using the naive series for the eigenvalues. It would be very interesting, however, to find a way to sum up our expansion in g and $\log(g)$ in order to calculate critical exponents at the nontrivial fixed point. This problem will be investigated in our future work.

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